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Generalizations of the Kerr–Newman and Charged Tomimatsu–Sato Metrics in the Jordan–Brans–Dicke Theory

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A procedure to construct stationary axisymmetric solutions of the Jordan-Brans-Dicke field equations with electromagnetic sources is obtained solutions, since they are "compositions" of Weyl static solutions with the given stationary ones, are equipped with several parameters, as many as one wishes.

1. INTRODUCTION

The main purpose of this work is to give the explicit expressions of certain families of stationary axisymmetric solutions (SAS) of the Jordan-Brans-Dicke (JBD) field equations coupled with the Maxwell (M) equations.

Acording to a theorem by García D. (1986), any SAS of the JBD+M field equations

$$\begin{aligned} R_{ab} &= 8\pi T_{ab}\phi^{-1} - \omega\phi^{-2}\phi_{;a}\phi_{;b} - \phi^{-1}\phi_{;a;b} \\ 4\pi T_{ab} &:= F^{s}_{a}F_{sb} + \tfrac{1}{4}g_{ab}F^{rs}F_{rs} \\ \phi^{;a}_{;a} &= 0 = F^{ab}_{;b} \end{aligned} \tag{1}$$

can be obtained by "composing" SAS of the Einstein-Maxwell (EM) equations with solutions of the static vacuum Weyl class. The metric for SAS of the JBD+M field equations is given by

$$\begin{aligned} g &= e^{-2U}\{f^{-1}[e^{2\tilde{\gamma}}(d\rho^2 + dz^2) + \rho^2\, d\psi^2] - f(dt - W\, d\psi)^2\} \\ \tilde{\gamma} &= \gamma + (3+2\omega)k, \qquad 2U := \ln \phi \end{aligned} \tag{2}$$

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with the set of structural functions f, γ, W and the associated electromagnetic vector potential field $A_\mu(\rho, z) = \delta^t_\mu A_t + \delta^\psi_\mu A_\psi$ is any solution of the standard Einstein–Maxwell (EM) field equations, while the set of U and k is any solution of the equations for static vacuum Weyl metric, namely,

$$\Delta U = U_{,\rho\rho} + \rho^{-1} U_{,\rho} + U_{,zz} = 0 \tag{3}$$

and

$$k = \int \{2\rho U_{,\rho} U_{,z}\, dz + \rho[(U_{,\rho})^2 - (U_{,z})^2]\, d\rho\} \tag{4}$$

this last relation being integrable by virtue of equation (3).

The Weyl solutions to be considered are the set of asymptotically flat Legendre polynominals solutions, and the $S(A, b, c/m)$ Plebański (1980) solutions. This last class of solutions contains as particular cases, among others, the Kasner (1921) metric $S(a, 0, 0/m)$ and the Zipoy (1966)–Voorhees (1970) solutions $S(0, \delta, 0/m)$.

In Section 2 we present a family of JBD + M solutions having as "seed" SAS the Kerr–Newman (KN) metric (Newman *et al.*, 1965).

In Section 3, SAS of the JBD+M theory having as seed metric the charged Tomimatsu–Sato (c-TS) solution (Ernst, 1973) are given.

2. A FAMILY OF JBD–KERR–NEWMAN METRICS

The Kerr–Newman solution in Weyl canonical coordinates in the Lewis–Papapetrou form is given by

$$g = f^{-1}\left[\frac{\Delta}{(r-m)^2 - m^2\cos^2\theta}(dz^2 + d\rho^2) + \rho^2\, d\psi^2\right]$$

$$-f\left(dt - a\sin^2\theta\,\frac{2mr - e^2}{\Delta}\, d\psi\right)^2$$

$$f = \frac{\Delta}{r^2 + a^2\cos^2\theta}, \qquad \Delta := (r-m)^2 + a^2\cos^2\theta + e^2 - m^2$$

$$A_\psi = -\frac{ear\sin^2\theta}{r^2 + a^2\cos^2\theta}, \qquad A_t = \frac{er}{r^2 + a^2\cos^2\theta},$$

$$M^2 := m^2 - e^2 - a^2 \tag{5}$$

where r and θ are the standard (r, θ) Boyer–Lindquist coordinates related to the (ρ, z) Weyl coordinates according to

$$\rho = \sin\theta[(r-m)^2 - M^2]^{1/2}, \qquad z = (r-m)\cos\theta \tag{6}$$

or

$$(r-m)^2 = \tfrac{1}{2}(\rho^2+z^2+M^2) \pm \tfrac{1}{2}[(\rho^2+z^2+M^2)^2 - 4M^2z^2]^{1/2}$$
$$\theta = \arccos[z/(r-m)], \qquad M^2 := m^2 - e^2 - a^2 \tag{7}$$

Comparing the metric (5) with the Lewis–Papapetrou one, i.e., the g from (2) with $U=0=k$, one readily obtains the explicit form of $\exp(2\gamma)$ and W, which are just the functions which multiply $(dz^2+d\rho^2)$ and $d\psi$, respectively.

We shall "compose" the KN metric (5), according to the rule given in the metric (2), with the set of asymptotically flat Legendre polynomial solutions.

The Legendre polynomial solutions arise as variable separable solutions of equations (3) written in spherical coordinates

$$\rho = R\sin\tilde{\theta}, \qquad z = R\cos\tilde{\theta} \tag{8}$$

[we are using $\tilde{\theta}$ here to distinguish it from the θ appearing in (5)-(7)].

The general class of asymptotically flat Legendre polynomial solutions (Kramer *et al.*, 1980) is given by

$$U = \sum_{n=0}^{\infty} a_n R^{-(n+1)} P_n(\cos\tilde{\theta})$$
$$k = -\sum_{l,m=0}^{\infty} a_l a_m \frac{(l+1)(m+1)}{(l+m+2)} R^{-(l+m+2)}(P_l P_m - P_{l+1}P_{m+1}) \tag{9}$$

where $P_s := P_s(\cos\tilde{\theta})$ are Legendre polynomials, and a_s are arbitrary parameters.

Combining the structural functions according to formulas (2), one obtains a class of JBD+M solutions with an infinite number of parameters; when they are equated to zero, the derived metric reduces to the standard Kerr–Newman metric.

3. A CLASS OF JBD–CHARGED-TOMIMATSU–SATO SOLUTIONS

In this section we derive a class of charged JBD solutions using as seed stationary axisymmetric metric the c-TS solution (Ernst, 1973) and as Weyl solutions the $S(a, b, c/m)$ Plebański (1980) metric.

To give in a concise form the c-TS solution, we introduce the following definitions

$$u^2 := x^2 - 1, \qquad v^2 := 1 - y^2, \qquad S\lambda := \sin\lambda, \qquad C\lambda := \cos\lambda, \qquad Q^2 := 1 - q^2 \tag{10}$$

The c-TS line element can be written as

$$g = f^{-1}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{u^2}+\frac{dy^2}{v^2}\right)+u^2v^2\,d\psi^2\right]-f(dt-W\,d\psi)^2 \quad (11)$$

with

$$\begin{gathered} f=A/B, \qquad W=4Cv^2S\lambda/(AQC\lambda) \\ \exp 2\gamma = A[(x^2-y^2)C\lambda]^{-4} \end{gathered} \quad (12)$$

where the functions A, B, and C are given by

$$\begin{aligned} A &= (u^4C\lambda^2+v^4S\lambda^2)^2 4u^2v^2(x^2-y^2)^2C\lambda^2S\lambda^2 \\ B &= (x^4C\lambda^2+y^4S\lambda^2-1+2xu^2C\lambda/Q)^2 \\ &\quad +4[x(x^2-y^2)C\lambda+v^2/Q]^2y^2S\lambda^2 \\ C &= \tfrac{1}{2}(Q+1/Q)\{u^2C\lambda^2[u^2v^2-4x^2(x^2-y^2)]+v^6S\lambda^2\} \\ &\quad -xC\lambda\{C\lambda^2u^2[2(x^4-1)+v^2(x^2+3)]-v^6S\lambda^2\} \end{aligned} \quad (13)$$

In these expressions λ and q are arbitrary real parameters associated, respectively, with the specific angular momentum and the charge-to-mass ratio.

The complex scalar electromagnetic potential is given by

$$\Phi(E) := A_t + i\mathscr{A} = qe^{i\alpha}D/(D+NQ) \quad (14)$$

where

$$\begin{aligned} N &= x^4C\lambda^2+y^4S\lambda^2-1-2ixy(x^2-y^2)C\lambda S\lambda \\ D &= 2xu^2C\lambda-2iyv^2S\lambda \end{aligned} \quad (15)$$

In formula (14), α is an arbitrary phase related to a duality rotation.

We point out that the electromagnetic field tensor and the stress energy tensor may be expressed entirely in terms of the Ernst complex potential ϕ, as was observed by Ernst (1974). In fact, in the Weyl coordinate chart $\{\rho, z, \psi, t\}$ related with the spheroidal prolate coordinates $\{x, y, \psi, t\}$ according to

$$\rho^2=(x^2-1)(1-y^2), \qquad z=xy, \qquad \psi=\psi, \qquad t=t \quad (16)$$

the nonvanishing components of the field tensor F_α^β are given by

$$\begin{aligned} F_{\dot z}^{t} &= -f^{-1}A_{t,z}+W\mathscr{A}_{,\rho}/\rho, \qquad & F_{\dot z}^{\psi} &= \mathscr{A}_{,\rho}/\rho \\ F_{\dot\rho}^{t} &= -f^{-1}A_{t,\rho}-W\mathscr{A}_{,z}/\rho, \qquad & F_{\dot\rho}^{\psi} &= -\mathscr{A}_{,z}/\rho \end{aligned} \quad (17)$$

Following the rule given in formula (2), we shall "compose" the above charged TS solution with the structural functions of the $S(a, b, c/m)$ metric.

The line element of the $S(A, b, c/m)$ solutions can be written as

$$m^{-2}g = e^{-2U}\left[e^{2k}(x^2-y^2)\left(\frac{dx^2}{u^2}+\frac{dy^2}{v^2}\right)+u^2v^2\,d\psi^2\right]-e^{2U}\,dt^2 \qquad (18)$$

where

$$\begin{aligned}\exp(2U) &= (x+1)^{-(a+b)}(x-1)^{b-a}(1+y)^{-(a+c)}(1-y)^{c-a}\\ \exp(2k) &= (x+y)^{-(b+c)2}(x-y)^{-(b-c)^2}(x+1)^{(a+b)^2}\\ &\quad\times(x-1)^{(a-b)^2}(1+y)^{(a+c)^2}(1-y)^{(a-c)^2}\\ u^2 &:= x^2-1, \qquad v^2 := 1-y^2\end{aligned} \qquad (19)$$

where a, b, and c are arbitrary dimensionless parameters, and m is a constant of length dimension.

Substituting all the required structural functions in the metric (2), one arrives at a certain class of JBD-charged-TS solutions. Particularly interesting is the set of solutions with the structural functions of the $S(0, \delta, 0/m)$ metric, i.e., the asymptotically flat Zipoy-Voorhees solutions, which reduce to the well-known Schwarzschild solution when $\delta = 1$.

Moreover, one may compose the c-TS solution with the asymptotically flat Legendre solutions given by formulas (9) simply by taking into account that

$$R^2 = (uv)^2 + (xy)^2, \qquad \tan\tilde{\theta} = (uv)^2/(xy) \qquad (20)$$

On the other hand, one may obtain a new class of JBD solutions by "composing" the Kerr–Newman metric (5) with the $S(a, b, c/m)$ solution. To do so, one needs to rewrite the structural functions of the $S(a, b, c/m)$ in terms of the ρ and z variables or give the KN solution in terms of the x and y variables. The relation between these sets of variables is given in (16). Notice that the Boyer–Lindquist r and θ are related to x and y according to $r - m = xM$, $\cos\theta = y$.

Since equation (3) for the function U is a linear one—the Laplace equation—one may linearly superpose any number of independent solutions, obtaining in this manner enlarged sets of solutions, for instance, the general Legendre polynomial (asymptotic and nonasymptotic) solutions coupled with the structural function U of the $S(a, b, c/m)$ or with the multiexponent Weyl U potential (Plebański and García, D., 1982). The structural function k in all cases will be certainly integrable.

4. CONCLUDING REMARKS

The procedure presented and used here can be applied to any stationary axisymmetric Einstein–Maxwell solution, in particular to static and vacuum ones, to generate an infinite class of solutions in the Jordan–Brans–Dicke theory. In doing so, one has to restrict oneself to solutions which may bear a physically relevant interpretation.

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